Pade and Rational Approximationt

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الخلاصة

استخدمنا في هذا العمل متعددات الحدود الاندراجية النسبية في التقريب و هذا ما يسمى تقريب بيد. يعتبر هذا العمل تعميم لمتعددات حدود تايلر .

الكلمات المفتاحية

تقريب بيد، تقريب تايلر .

Abstract

In this paper we shall introduce a class of interpolating rational functions called pade approximants. These rationales provide a natural extension of the Taylor sections.

Keywords

Pade approximants, Taylor sections.

1.Introductin and Preliminaries

We take a formal power series

$f(z)=\sum_{k=o}^{\infty} a_k z^k$ (1.1)

We shall construct rational function of a certain type whose Taylor coefficients match those of f. Let

$Л_{m,n} ;=\left\{R(z)=\frac{S(z)}{T(z)}: S\in Л_m , Q \in Л_n , T\neq 0\right\}$ (1.2)

Then for a fixed pair (m,n), we can find an $R(z)\in Л_{m,n}$, such that $(f-R)(z)=O(z^l)$, where l approaches ∞ . (1.3)

$O(z^l)$ denotes a power series with lowest order term z^l.what is a realistic value for l? Because there are m+1 free parameters in the choice for the number p, and n+1 in choice for the denominator Q, then there are m+n+1 parameters in the ration S/T (one parameter is lost in the division process. Then we have l ≥ m+n+1 or, equivalently, to match the first m+n+1 terms of (1.1). and this is not possible (try m=0,n=1, and f(z)=z). We instead above with the following linearized version of (1.3). Take (m,n), $p_{mn}\in Л_n$ and Q_{mn} (≠o)$\in Л_n$ so that

$(Q_{mn} f - p_{mn})(z)=O(z^{m+n+1})$. (1.4)

If f is (m+n) - times differentiable at z=0, then(1.4) is equal to $(Q_{mn}\ f-p_{mn})(0)=0$, k=o,1,… ,m+n .That (1.4) represents a homogeneous system of m+n+1 equations in m+n+2 unknowns (the coefficients of p_{mn} and Q_{mn}). Hence this system has a non trivial solution, necessarily with Q_{mn}≠o. We shall give

Definition (1.5): [1] The pade approximant (PA) of type (m,n) to f is the rational

$\left[\frac{m}{n}\right](z) := \frac{S_{mn}(z)}{T_{mn}(z)}$, (1.6)

where $S_{mn}\in Л_m$ and ≠0 $T_{mn}\in Л_n$ satisfy (1.4).

Notice that for n=0, the PA reduces to a Taylor section of (1.1).

$\left[\frac{m}{0}\right](z) = \sum_{k=0}^{m} a_k z^k$. (1.7)

$(R_1 f - S_1)(z)=O(z^{m+n+1})$, $(R_2\ f - S_2)(z) = O(z^{m+n+1})$. (1.8)

$(R_1 S_2 - R_2 S_1) = O(z^{m+n+1})$. (1.9)

Notation: Let $Л_n$ denote the collection of polynomials of degree ≤ n, R is (n,m) rational function i. e R= S/T where S ∈ $Л_n$ and ∈ $Л_m$. If f is analytic in some neighborhood U of zero then the (n,m) pade approximation to f is an (n,m) rational function R=S/T that satisfies

$S(z)-T(z)f(z)=z^{n+m+1} g(z)$ (1.10)

where g(z) is analytic in U, and T(z) is not identically zero. We will call R proper if T(o)≠0, that is, if g is analytic in some neighborhood of zero the proper (n,m) pade approximation is unique

Let $\|.\|_{I_{p(E)}}=\left(\int_E |f(z)|^p\, dz\right)^{\frac{1}{p}}$, $0 < p < \infty$

where $I_{p(E)}=\{f:\|f\|_{I_{p(E)}}=\|f\|_p < \infty\}$

And we say that an (n,m) rational function R is a best (n,m) rational approximation to f in $I_{p(E)}$ if R satisfy

$\|f-R\|_{I_{p(E)}} = \int_E |f(z) - S/T|^p\, dz)^{\frac{1}{p}}$, $0<p<\infty$, where $\mathbb{C}\supset E$.

Let $U_s=\{z\in C:|z|<\delta\}$, $C_\delta^{\prime}=\{z\in C: |z|=\delta\}$, δ > 0

The PA is unique. In fact: $(R_1 f - S_1)(z)= O(z^{m+n+1})$ and $((R_2 f - S_2)(z) = O(z^{m+n+1})$, where $S_1,S_2\in Л_m$ and $T_1,T_2\in Л_n$. Multiplying the equation in (1.8) by R_2 and the second by R_1,Then we have $(R_1 S_2 - R_2 S_1) = O(z^{m+n+1})$. But the left hand side of (1.9) is a polynomial of degree ≤ m + n. Hence $R_1 S_2 - R_2 S_1= 0$ or $S_1/T_1 = S_2/T_2$.

2. The Main Results

In this section we shall introduce our main results. First we want to mention that monic polynomial is polynomial of leading coefficient equal to 1.

2.1. Theorem

Let f be analytic in the disk K:|z|<R . (0<R≤∞) except for poles of total multiplicity v, none of which occurs at z=0 . Then as m→∞, the sequence of pade approximants [m/r](z) converge to f(z) in L_p (U) on every compact subset E of U⁄{poles of f}.

Proof:

since [m,v](z) is pade approximants to f so R_{mv} $f-S_{mv})(z)= O(z^{m+v+1})$.

If R is a monic polynomial of degree ≤ v, then RS_{mv} is a polynomial of degree ≤ m + v, with

$R_{mv}\ Rf - RS_{mv} = O(z^{m+v+1})$

This implies RS_{mv} is the Taylor polynomial of the function R_{mv} Rf.

Then using $f(z)-S_n(z)=\frac{1}{2\pi i}\int_{|t|=r}\frac{z^{n+1}f(t)}{t^{n+1}(t-z)}dt, |z|<r$ to get

R_{mv} Rf is analytic on |t| ≤ r.

Since R is monic then its zeros are the poles of f and this implies r≅R .

$\|R_{mv}\ Rf - RS_{mv}\|_{I(p(K)/(\text{poles } f)}$

$=(\int|R_{mv}\ Rf - RS_{mv}|^p\, d z)^{1/p}$

$= (\int_{c/poles\ f}\left|\frac{1}{2\pi i}\int_{|t|=r} z^{m+v+1} R_{mv}\, Rf(t)\right|^p$ ♦

2.2. Corollary

Let f be analytic in $I_p(U_s)$ and it has a proper (n,m) pade approximation. Let R be a best (n,m) rational approximation to f on $I_{p(U)}$, Then $\|f(z)-R(z)\|_{I_{p(C_\delta^{\prime})}} = \|f(z)-P(z)+P(z)-R(z)\|_{I_{p(C_\delta^{\prime})}}$

Proof:

Suppose $\|f(z)-R(z)\|_{I_{p(C_\delta^{\prime})}}<\|f(z)-P(z)\|_{I_{p(C_\delta^{\prime})}}$

As a corollary of the well known theorem (Let E be origin bounded by the Jordan curve Γ, f∈I_p (E), and $P_n\in Л_n$. If the error curve $(f-P_n)(Γ)$ is a perfect circle of center at the origin and winding number ≥ n + 1. Then P_n is the best I_p (E) approximation to f with degree ≥ n + 1)

we get :

$\|f(z) - R(z)\|_{I_{p(C_\delta^{\prime})}} = \|f(z) - P(z) + P(z) - R(z)\|_{I_{p(C_\delta^{\prime})}}$

2.3. Lemma [3]

If s/t be the (n,n) pade approximation of e^z then

a- $e^z-\left(\frac{s(z)}{t(z)}\right)=\frac{(-1)^n z^{2n+1}}{t(z)(\prod 2n)}\int_0^1 e^{tz}\,(1-t)^n\, t^n\, dt$

b- $t(z)=\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\,\frac{(-z)^k}{\prod k}$

c- $2-e^{\frac{1}{2}}\leq |\,t(z)|\ \leq e^{\frac{1}{2}}$

d- $\int_0^1 e^{-t}(\,(\cos t)\,(1-t)^n\, t^n\, dt)\geq\left(\frac{(e^{-\frac{1}{2}}\cos\frac{1}{2}+e^{-1}\cos 1)}{2}\right)\frac{\prod(n)^2}{\prod(2n+1)}$.

2.4. Theorem

Assume R_n be a best (n,n) rational approximation of e^z . defined in $C_1^{\prime}$ and P_n is the (n,n) pade approximation of e^z Defined in $C_1^{\prime}$.Then $\|R(z)-e^z\|_{I_{p(C_1^{\prime})}}\geq\|P_n(z)-e^z\|_{I_{p(C_1^{\prime})}}$

$$\left\|e^z-\left(\frac{s(z)}{t(z)}\right)\right\|_{I_{p(C_1^{\prime})}}\leq\frac{(2\pi)^{\frac{1}{p}}\ e}{\left(2-e^{\frac{1}{2}}\right)\prod(2n)} \qquad (2.5)$$

And

$$\left\|e^z-\left(\frac{s(z)}{t(z)}\right)\right\|_{I_{p(C_1^{\prime})}}\geq\frac{(2\pi)^{\frac{1}{p}}}{e^{\frac{1}{2}}\prod(2n)}\left(\frac{(e^{-\frac{1}{2}}\cos\frac{1}{2}+e^{-1}\cos 1)}{2}\right)\frac{(\prod n)^2}{\prod(2n+1)} \qquad (2.6)$$

Proof:

Let P_n = S/T is the (n,n) pade approximation to e^z

$$\left\|\int_0^1 e^{tz}\,(1-t)^n\, t^n\, dt\right\|_{I_{p(C_1^{\prime})}}$$

$$\left(\int_{C_1^{\prime}}\left(\int_0^1 e^{tz}\,(1-t)^n\, t^n\, dt\right)^p dz\right)^{1/p}$$

$$\leq\left(\int_{C_1^{\prime}}\left(\int_0^1 e^{t}\,(1-t)^n\, t^n\, dt\right)^p dz\right)^{1/p}$$

$$=(2\pi)^{1/p}\int_0^1 e^{t}\,(1-t)^n\, t^n\, dt$$

$$\leq(2\pi)^{\frac{1}{p}}\int_0^1 e^{t} = (2\pi)^{\frac{1}{p}}\int_0^1\sum_{k=0}^{\infty}\frac{t^k}{\prod k}\,dt$$

$$=(2\pi)^{1/p}\int_0^1\sum_{k=0}^{\infty}\frac{1}{k} = (2\pi)^{1/p}\, e$$

So

$$\left\|\int_0^1 e^{tz}\,(1-t)^n\, t^n\, dt\right\|_{I_{p(C_1^{\prime})}}\leq(2\pi)^{1/p}\, e \qquad (2.7)$$

Then

$$\left\|\int_0^1 e^{tz}(1-t)^n t^n\,dt\right\|_{I_{p(C_1)}}$$

$$\geq (2\pi)^{\frac{1}{p}}\left(\frac{(e^{-\frac{1}{2}}\cos\frac{1}{2}+e^{-1}\cos 1)}{2}\right)\frac{(\prod n)^2}{\prod(2n+1)} \quad (2.8)$$

Therefor

Using Lemma 2.3 we get

$$\left\|e^z-\left(\frac{s(z)}{t(z)}\right)\right\|_{I_{p(C_1)}} = \left\|\frac{(-1)^n z^{2n+1}}{t(z)\prod(2n)}\int_0^1 e^{tz}(1-t)^n t^n\,dt\right\|_{I_{p(C_1)}}$$

Then by d of 2.3. Lemma we get

$$\left\|e^z-\left(\frac{s(z)}{t(z)}\right)\right\|_{I_{p(C_1)}} = \left(\int_{C_1}\left|\frac{(-1)^n z^{2n+1}}{t(z)\prod(2n)}\int_0^1 e^{tz}(1-t)^n t^n\,dt\right|^p dz\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{(2-e^{1/2})(2n)}\left\|\int_0^1 e^{tz}(1-t)^n t^n\,dt\right\|_{I_{p(C_1)}}$$

Then by (2.7) we obtain

$$\left\|e^z-\left(\frac{s(z)}{t(z)}\right)\right\|_{I_{p(C_1)}} \leq \frac{(2\pi)^{\frac{1}{p}}\,e}{\left(2-e^{\frac{1}{2}}\right)\prod(2n)} \quad (2.5)$$

$$\left\|e^z-\left(\frac{s(z)}{t(z)}\right)\right\|_{I_{p(C_1)}} \geq \frac{1}{e^{\frac{1}{2}}\prod(2n)}\left\|\int_0^1 e^{tz}(1-t)^n t^n\,dt\right\|_{I_{p(C_1)}}$$

Then by (2.8) we have

$$\left\|e^z-\left(\frac{s(z)}{t(z)}\right)\right\|_{I_{p(C_1)}} \geq \frac{(2\pi)^{\frac{1}{p}}}{e^{\frac{1}{2}}(2n)}\left(\frac{(e^{-\frac{1}{2}}\cos\frac{1}{2}+e^{-1}\cos 1)}{2}\right)\frac{(n)^2}{(2n+1)} \;❦$$

2.5. Lemma [2]

If $(z)=\int_0^1\frac{d\alpha(t)}{1+zt}$, where α(t) is a real non-decreasing function assuming infinitely many values on [0,γ]. Such functions are called Stieltjes series . And if P is the (n,n) Pade approximation to f then

$$f(z)-P(z)=\frac{1}{P_n^2\left(-\frac{1}{z}\right)}\int_0^1\frac{d\alpha(t)}{1+zt}$$

where $z\in\mathbb{C}-(-\infty,-\frac{1}{\gamma}]$ and P_n is a real polynomial 0f degree n with roots in [o,γ] .

2.6. Theorem

Suppose $f(z)=\int_0^\gamma\frac{d\alpha(t)}{1+zt}$ $,0<\gamma<1$, α is non-decreasing has many infinitely values on [o,γ]. If P is (n-1,n) Pade approximation to f defined on the circle C_1. Then

$$\|f(z)-R(z)\|_{I_{p(C_1)}} \geq \frac{(1-\gamma)^{2n}}{2^{1/p}\,(1+\gamma)^{2n}}\|f(z)-P(z)\|_{I_{p(C_1)}} \quad (2.11)$$

Proof:

Let P_n be a real polynomial as defined in Lemma (2.9). Therefore

$$\|f(z)-P(z)\|_{I_{p(C_1)}} = \left(\int_{C_1}\left|P_n^2\left(-\frac{1}{z}\right)\int_0^\gamma\frac{P_n^2\,d\alpha(t)}{1+zt}\right|^p dz\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{m}\left(\int_{C_1}\left|\int_0^\gamma\frac{P_n^2\,d\alpha(t)}{1}\right|^p dz\right)^{\frac{1}{p}} = \frac{1}{m}\left|\int_0^\gamma\frac{P_n^2\,d\alpha(t)}{1}\right|^p\left(\int_{C_1}dz\right)^{\frac{1}{p}}$$

$$=\frac{E}{m}(2\pi)^{\frac{1}{p}} \quad (2.12)$$

where $E=\int_0^\gamma P_n^2\,d\alpha(t)$, $m=\min_{z\in C_1}\left\{P_n^2\left(-\frac{1}{z}\right)\right\}$

$$\|f(z)-P(z)\|_{I_{p(C_1)}} = \left(\int_{C_1}\left|\frac{1}{P_n^2\left(-\frac{1}{z}\right)}\int_0^\gamma\frac{P_n^2\,d\alpha(t)}{1+zt}\right|^p dz\right)^{\frac{1}{p}}$$

$$\geq \frac{E}{M}\left(\int_{C_1}\left|\frac{1}{1+z\gamma}\right|^p dz\right)^{\frac{1}{p}}$$

$$\geq \frac{E}{M}\left(\int_{C_1}\frac{1}{1+1}dz\right)^{\frac{1}{p}}$$

$$=\frac{E}{M}(\pi)^{\frac{1}{p}} \quad (2.13)$$

where $M=\max_{z\in C_1}\left\{P_n^2\left(-\frac{1}{z}\right)\right\}$. Since $P_n^2\left(-\frac{1}{z}\right)$ has all roots in $(-\infty,-\frac{1}{\gamma}]$ so that

$$\frac{M}{m}\leq\frac{\left(\frac{1}{\gamma}+1\right)^{2n}}{\left(\frac{1}{\gamma}-1\right)^{2n}}=\left(\frac{1+\gamma}{1-\gamma}\right)^{2n}.$$

Then by (2.12) and(2.13) we get

$$\frac{\frac{E}{m}(2\pi)^{\frac{1}{p}}}{\frac{E}{M}(\pi)^{\frac{1}{p}}} = \frac{2^{\frac{1}{p}}M}{m} \leq \frac{2^{\frac{1}{p}}(1+\gamma)^{2n}}{(1-\gamma)^{2n}}$$

Therefore Corollary 2.2 implies (2.11) ❦

2.7. Theorem

Let $f\in A_{\tilde{p}}$ and suppose that $\frac{P_n}{q_n}$ is the proper (n,m) pade approximation to f. Let $\gamma<\rho<\tilde{\rho}$ and let $\frac{S_n}{T_n}$ be a best rational approximation to f on $\{z:|z|\leq\rho\}$. If $\frac{P_n}{q_n}$ has no poles in D_ρ then for $|z|<\gamma$,

$$\left|f(z)-\frac{P_n(z)}{q_n(z)}\right| \leq \frac{|z|^{m+n+1}}{\gamma^{m+n}}\left(1+\frac{2\gamma}{\rho-\gamma}\right)^{2m}\frac{\left\|f(z)-\frac{S_n(z)}{T_n(z)}\right\|_{c_\gamma}}{\gamma-|z|}$$

Proof:

$$\left\|f(z)-\frac{P_n(z)}{q_n(z)}\right\|_{I_p(c_\gamma)} =$$

$$\left(\left|\int_{c_\gamma}\frac{z^{m+n+1}}{|q_m(z)T_m(z)|}\;\frac{\frac{1}{2\pi i}\int_{c_\gamma}q_m(\aleph)(T_m(\aleph)f(\aleph)-S_n(\aleph))d\aleph}{\aleph^{m+n+1}}\right|^p dz\right)^{\frac{1}{p}}$$

$$\leq\left(\int_{D_\gamma}\left(\frac{|z|^{m+n+1}}{|q_m(z)T_m(z)|}\frac{1}{2\pi i}\int_{c_\gamma}\frac{q_m(\aleph)(T_m(\aleph)\;(f(\aleph)-(\frac{S_n(\aleph)}{T_m(\aleph)}))d\aleph}{(\gamma-|z|)\,(\gamma^{m+n})}\right)^p dz\right)^{\frac{1}{p}}$$

Let $M=\max_{c\gamma}|q_m(\aleph)\,T_m(\aleph)|$

$$m=\min_{D_\gamma}|q_m(z)T_m(z)|$$

$$\left\|f(z)-\frac{P_n(z)}{q_n(z)}\right\|_{I_p(c_\gamma)} \leq$$

$$\left(\int_{D_\gamma}\left(\frac{\gamma^{m+n+1}}{m}\frac{1}{2\pi i}\frac{M}{1}\;\frac{1}{\gamma^{m+n}}\int_{c_\gamma}(f(\aleph)-(\frac{S_n(\aleph)}{T_m(\aleph)})d\aleph\right)^p dz\right)^{\frac{1}{p}}$$

$$=\frac{\gamma\;M}{m\;2\pi i}\left(\int_{D_\gamma}dz\right)^{\frac{1}{p}}\left(\left(\int_{c_\gamma}\left|f(\aleph)-(\frac{S_n(\aleph)}{T_m(\aleph)})\right|d\aleph\right)^p\right)^{1/p}$$

$$=\frac{\gamma\;M}{m\;2\pi i}\;(\gamma^2\pi)^{1/p}\;\left\|f(\aleph)-(\frac{S_n(\aleph)}{T_m(\aleph)})\right\|_{I_p(c_\gamma)}$$

Using definition of the rational approximation and the fact that any two norms of the space of polynomials are equivalents we complete the proof ❦

3. Some Essential Difference Between Polynomial and Rational Approximation on the Complex plane

We introduce some essential difference between polynomial approximation and rational approximation.

3.1. Convergence

In rational approximation we have if f is analytic on a compact set E not separate the complex plane {ℂ/E connected} then f is the I_p limit on E of a sequence of rational functions. Unlike its polynomial version the hypothesis ℂ/E be connected is not needed .

3.2. Existence of best approximation

For an arbitrary compact set the existence of best polynomial approximation is a simple compactness argument. Unlike its rational version needs that the compact set contains no isolated points.

3.3. Uniqueness of best approximation

If f is a real valued continuous function defined on the interval [a,b]. Then Chebyshev showed that the best uniform approximation R to f out of $Л_{m,n}$ is unique if R is of real coefficients.

3.4. Degree of best approximation

If f is continuous on E and analytic on the E interior. Let E_n (f)=$\inf_{p\in Л_n}$ $\|f-p\|_E$ be the degree of best polynomial approximation to. And e_n (f)=$\inf_{R\in Л_{n,m}}$ $\|f-R\|_E$ is the degree of rational approximation of. Since e_n (f)≤E_n (f) so e_n (f) tend to zero faster than E_n (f).

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